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THERE ARE ONLY FINITELY MANY FINITE DISTANCE-TRANSITIVE GRAPHS OF GIVEN VALENCY GREATER THAN TWO

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The result of the title is proved, assuming the truth of Sims' conjecture on primitive permutation groups (which has recently been established using the classification of finite simple groups). An alternative approach to this result, using less group theory but relying on a theorem of Macpherson on infinite distance-transitive graphs, is explored.

1. Introduction

A connected graph Γ is *distance-transitive* if, for any two pairs (x_1, x_2) and (y_1, y_2) of vertices with $d(x_1, x_2)=d(y_1, y_2)$ (where d is the distance in Γ), some automorphism of Γ maps x_i to y_i for $i=1, 2$. The finite distance-transitive graphs of valency 3 were determined by Biggs and Smith [2]: there are exactly twelve of them, up to isomorphism. On the basis of this result, it is natural to conjecture that, for any $k \geqq 3$, there are only finitely many finite distance-transitive graphs of valency k. This was proved for $k=4$ by Smith [7, 8, 9] (who found all such graphs). In both results, the heart of the argument involved converting a bound for the order of the group of automorphisms stabilising a vertex into a bound for the diameter (and hence for the number of vertices).

In this note, I show how to carry out Smith's program. I also observe that the required bound for the order of the vertex-stabiliser follows from the conjecture of Sims on primitive permutation groups, which has recently been proved [3], using the classification of the finite simple groups. In other words, the following holds, *assuming that all finite simple groups are known.*

Theorem. *There are only finitely many finite distance-transitive graphs of given valency $k>2$ (up to isomorphism).*

Finally, I indicate how the determination of infinite distance-transitive graphs of finite valency by Macpherson [5] raises the possibility of a more elementary proof. The Macpherson theorem, together with a compactness argument, reduces the proof of the theorem to a problem on local arc-transitivity in finite graphs, which might be settled by group-theoretic methods much less sophisticated that those required in the proof of the classification of finite simple groups.

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2. Proof of the theorem

The conjecture of Sims referred to in the Introduction asserts the existence of a function f such that, if G is a primitive permutation group on a set X in which the stabiliser G_x of a point $x \in X$ has an orbit of size k (other than $\{x\}$), then G_x has order at most $f(k)$. This has recently been proved, using the classification of finite simple groups, by Cameron, Praeger, Saxl and Seitz [3]. Until the appearance of that paper and the general acceptance of the classification theorem, the argument which follows can be regarded as conditional on Sims' conjecture, in the spirit of Smith's pioneering work on this problem.

We show first that, if the distance-transitive graph Γ of valency $k>2$ has a vertex-primitive automorphism group, then the diameter of Γ is at most $8f(k)+11$. Suppose, for a contradiction, that such a graph has diameter at least $8f(k)+12$.

Let c_i, a_i, b_i be the intersection numbers as defined by Biggs [1]: if $d(x, y)=i$, then the number of vertices adjacent to x and distant $i-1$, i, $i+1$ from y are c_i, a_i; b_i respectively. If k_i is the number of vertices distant i from x, then $k_i b_i = k_{i+1} c_{i+1}$. The sequence $\{b_i\}$ is nonincreasing, and the sequence $\{c_i\}$ is nondecreasing; so $\{k_i\}$ is unimodal, increasing for e_1 steps, remaining constant for e_2 steps, and decreasing for e_3 steps, where $e_1+e_2+e_3=d$, the diameter of Γ. Since each k_i is an orbit lenght for G_x, we have $k_i \leqq f(k)$, and so $e_1, e_3 \leqq f(k)$. Thus $e_2 \geqq d-2f(k) \geqq 6f(k)+12$. Note that c_i, a_i and b_i are constant for $e_1+1 \leqq i \leqq e_1+e_2-1$.

Select a point x. Set $\Omega=\{y \mid e_1+1 \leqq d(x, y) \leqq e_1+e_2-1\}$; and, for any $y \neq x$, set $S(y)=\{z \mid z \sim x, d(z, y)=d(x, y)-1\}$. (We write $x \sim y$ to denote "x and y are adjancent".) Now, if $y, y' \in \Omega$ and $y \sim y'$, then $S(y)=S(y')$. For, setting $i=d(x,y)$, there are three cases:

(i) $d(x, y')=i-1$; then $S(y') \subseteqq S(y)$ and $|S(y')|=c_{i-1}=c_i=|S(y)|$.

(ii) $d(x, y')=i+1$: then $S(y') \supseteqq S(y)$ and $|S(y')|=c_{i+1}=c_i=|S(y)|$.

(iii) $d(x, y')=i$. For any $z \in S(y)$, we have

$$\{w \mid w \sim y,\ d(w, z) \leqq i-1\} \subseteqq \{w \mid w \sim y,\ d(w, x) \leqq i\};$$

but the cardinalities $c_{i-1}+a_{i-1}$ and c_i+a_i of these sets are equal, so the sets themselves are equal. Similarly,

$$\{w \mid w \sim y,\ d(w, z) < i-1\} = \{w \mid w \sim y,\ d(w, x) < i\},$$

whence

$$\{w \mid w \sim y,\ d(w, z) = i-1\} = \{w \mid w \sim y,\ d(w, x) = i\}.$$

The second set contains y', and so the first does also. This holds for any such z; so $S(y') \supseteqq S(y)$, whence $S(y')=S(y)$ as before. (The argument fails if $i=e_1+1$; in that case, simply reverse all inclusions and inequalities in the three displayed formulae and replace c_{i-1}, c_i by b_{i-1}, b_i.)

Now it follows that, if y and y' lie in the same connected component of the induced subgraph on Ω, then $S(y)=S(y')$. Let b be the number of conneted components, and c the (constant) number of vertices in each component at distance i from x, so that $bc=k_i$ (for $e_1+1 \leqq i \leqq e_1+e_2-1$). Put $j=[\frac{1}{2}e_2]-2 \geqq 3f(k)+4$. Choose a point y at distance e_1+2+j from x, and let Δ be the connected component of Ω

containing y. If $B_i(x)$ denotes the closed ball of radius i centred at x, we have

$$B_j(y) \subseteqq \Delta \cap (B_{e_1+2+2j}(x) \setminus B_{e_1+1}(x)),$$

so $|B_j(y)| \leqq (2j+1)c$. But on the other hand

$$|B_j(y)| = |B_j(x)| = |B_{e_1+1}(x)| + (j-e_1-1)bc.$$

This gives $2j+1 > (j-e_1-1)b$, or $b < (2j+1)/(j-e_1-1)$. Since $j \geqq 3e_1+4$, we have $b<3$, whence $b \leqq 2$.

The group G_x acts transitively on $\Gamma(x)$, and permutes among themeselves the (at most two) distinct sets $S(y)$ for $y \in \Omega$; so either $S(y) = \Gamma(x)$ for all $y \in \Omega$, or there are two distinct sets $S(y)$ which are blocks of imprimitivity for G_x in $\Gamma(x)$. Since $|S(y)| = c_i$ for $d(x, y) = i$, the first is impossible, and we have $c_i = \frac{1}{2}k$ for $e_1+1 \leqq i \leqq e_1+e_2-1$. Now $k_i b_i = k_{i+1} c_{i+1}$, so $b_i = \frac{1}{2}k$, $a_i = 0$. It follows that Ω is bipartite.

Since Γ is vertex-primitive, it is not bipartite, and it contains a cycle of odd length $2s+1$. Choose such a cycle containing x with s minimal. We show that $s \leqq e_1$. If not, then clearly $s \geqq e_1+e_2$. If y and y' are the two vertices of the cycle at distance s from x, then $S(y) \cap S(y') = \emptyset$, since otherwise there is a shorter odd cycle. But $|S(y)| = c_s \geqq c_{e_1+1} = \frac{1}{2}k$; so $|S(y)| = \frac{1}{2}k$. We have $c_i = \frac{1}{2}k$ and $a_i = 0$, whence $b_i = \frac{1}{2}k$, for $e_1+e_2 \leqq i \leqq s-1$; by definition of e_2, we have $s = e_1+e_2$.

If u is the vertex of the cycle adjacent to x and distant $s-1$ from y', then of the k neighbours of y, $\frac{1}{2}k$ are distant $s-1$ from x, and the other $\frac{1}{2}k$ are distant $s-1$ from u (and hence s from x). Thus $a_s = \frac{1}{2}k$, $b_s = 0$, and the graph has diameter s. Now put $K_0 = \{x\}$, $K_i = \{z \mid d(x, z) = i,\ S(z) \subseteqq S(y)\}$, and $K_{-i} = \{z \mid d(x, z) = i,\ S(z) \subseteqq S(y')\}$ for $1 \leqq i \leqq s$. (Thus $K_1 = S(y)$, $K_{-1} = S(y')$.) Then any edge of the graph joins a vertex of K_i to a vertex of K_{i-1} for some i, where the subscripts are taken modulo $2s+1$. So every $(2s+1)$-cycle contains x, an obvious contradiction, since the graph is not a cycle.

So $s \leqq e_1$, as claimed. Now a $(2s+1)$-cycle through a vertex at distance e_1+s+1 from x lies wholly within Ω (since $e_2 \geqq 2s+2$), contradicting the fact that Ω is bipartite.

This concludes the argument in the primitive case.

Now let Γ be any distance-transitive graph of valency $k>2$, and let G be its automorphism group. If G is imprimitive on vertices then, by a theorem of Smith [6], Γ is either bipartite or antipodal. If it is antipodal, and $\tilde{\Gamma}$ is the graph obtained by identifying antipodal vertices, then $\tilde{\Gamma}$ has valency k and is distance-transitive but not antipodal; and $d \leqq 2\tilde{d}+1$, where d and $\tilde{d}$ are the diameters of Γ and $\tilde{\Gamma}$ respectively. If Γ is bipartite, let Γ_2 be the graph whose vertex set is a bipartite block of Γ, two vertices adjacent whenever they lie at distance 2 in Γ. Then Γ_2 is a distance-transitive graph of valency at most $k(k-1)$, and not bipartite (since it contains a triangle); with a similar notation, $d \leqq 2d_2+1$. Furthermore, if Γ_2 is antipodal, then $\tilde{\Gamma}_2$ is neither bipartite nor antipodal, hence $\tilde{\Gamma}_2$ is vertex-primitive. We conclude that the diameter of Γ is bounded.

The preceding argument shows, in effect, that if there were infinitely many finite distance-transitive graphs of valency $k>2$, then there would be infinitely many vertex-primitive distance-transitive graphs of some valency $k'>2$. We will use this fact in the next section.

3. A compactness argument

Macpherson [5] has determined all infinite distance-transitive graphs of finite valency k. In such a graph Γ, the neighbourhood of a vertex is a disjoint union of $t+1$ complete graphs of size s, where s and t are positive integers with $s(t+1)=k$; so the maximal cliques in Γ have size $s+1$, and any vertex lies in $t+1$ of them. Moreover, the number of vertices at distance i from a given vertex is $(t+1)t^{i-1}s^i$, so that there are no cycles apart from those contained in cliques.

The statement that Γ is a graph of valency k and G a group acting on Γ as a distance-transitive automorphism group can be expressed (by infinitely many sentences) in a suitable first-order language. So any first-order property which holds in infinitely many such finite graphs also holds in an infinite distance-transitive graph, by the Compactness Theorem. The structure of the neighbourhood of a vertex, and the number of vertices at given distance from a vertex, are candidates for such properties. We conclude that, if there are infinitely many finite distance-transitive graphs of valency k, then there is a factorisation $k=(t+1)s$ such that, for any integer $d>0$, a finite distance-transitive graph Γ_d exists with the properties

(i) the neighbourhood of a vertex consists of a disjoint union of $t+1$ complete graphs of size s;

(ii) for $i\leq d$, the number of vertices distant i from a given vertex is $(t+1)t^{i-1}s^i$.

As remarked at the end of the last section, we may also assume that Γ_d has vertex-primitive automorphism group.

Now the order of the stabilisier of a vertex in Γ_d is divisible by $(t+1)t^{d-1}s^d$. Of course the truth of Sims' conjecture shows that this is impossible. However, we would like to derive a contradiction using more elementary group theory.

The first positive result on Sims' conjecture was a theorem of Thompson [10] asserting that, with the same hypotheses, G_x has a normal subgroup whose order is a prime power and whose index is bounded by a function of k. In our case, choosing d sufficiently large, we see that both s and t are powers of the Thompson prime p. (Of course, one of s and t may be equal to 1.) We remark that at this point the theorem is proved (in the primitive case) for almost all k, viz. those not expressible in the form p^a+p^b with p prime.

Define a new graph Γ_d^* as follows. Vertices of Γ_d^* are of two types, the vertices and maximal cliques of Γ_d; an edge of Γ_d^* joins vertices of different types whenever they are incident in Γ_d. Then Γ_d^* is bipartite and semiregular, with valencies $s+1$ and $t+1$; and it is clear that Γ_d^* is locally $(2d-1)$-arc transitive (i.e. the stabiliser of a vertex acts transitively on the $(2d-1)$-arsc commencing at that vertex.) So it will be enough to find a bound, in terms of s and t, for the degree of local arc-transitivity of such a graph. There is some evidence that this can be done directly; and there may even be an absolute bound (independent of s and t). (For example, Gardiner [4] showed that a graph of valency $p+1$, for p prime, can be at most 7-arc transitive, while Weiss [11] showed that a cubic graph is at most locally 7-arc transitive.)

Added in prof. The equation $k_i b_i = k_{i+1} c_{i+1}$ shows that if $k_{i+1} > k_i$, then $k_{i+1} \geqq (k/k-1)k_i$. Using this, the inequalities e_1, $e_3 \leqq f(k)$ can be improved to e_1, $e_3 \leqq k \log f(k)$, giving an improved bound $8k \log f(k) + 11$ for the diameter of a primitive distance-transitive graph.

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